**RELATIVISTIC COSMOLOGY**

## 3.1 The Robertson-Walker Metric

The appearance of objects at cosmological distances is affected by the curvature of spacetime through which light travels on its way to Earth. The most complete description of the geometrical properties of the Universe is provided by Einstein's general theory of relativity. In GR, the fundamental quantity is the metric which describes the geometry of spacetime.

Let's look at the definition of a metric: in 3-D space we measure the distance along a curved path $\mathcal{P}$ between two points using the differential distance formula, or *metric*:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{3.1}$$

and integrating along the path $\mathcal{P}$ (a line integral) to calculate the total distance:

$$\Delta\ell = \int_1^2 \sqrt{(d\ell)^2} = \int_1^2 \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{3.2}$$

Similarly, to measure the interval along a curved wordline, $\mathcal{W}$, connecting two events in spacetime with no mass present, we use the metric for flat spacetime:

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{3.3}$$

Integrating $ds$ gives the total interval along the worldline $\mathcal{W}$:

$$\Delta s = \int_A^B \sqrt{(ds)^2} = \int_A^B \sqrt{(cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2} \tag{3.4}$$

By definition, the distance measured between two events, $A$ and $B$, in a reference frame for which they occur simultaneously ($t_A = t_B$) is the proper distance:

$$\Delta\mathcal{L} = \sqrt{-(\Delta s)^2} \tag{3.5}$$

Our search for a metric that describes the spacetime of a matter-filled universe, is made easier by the cosmological principle. In a homogeneous and isotropic universe, although the curvature of space may change with time, it must have the same value everywhere at a given time since the Big Bang.

On the surface of a sphere, curvature is defined as $K \equiv 1/R^2$. But a more general expression for curvature in a 2-D space is (see Figure 3.1)

$$K = \frac{3}{\pi} \lim_{D \to 0} \frac{2\pi D - C_{\rm meas}}{D^3} \tag{3.6}$$

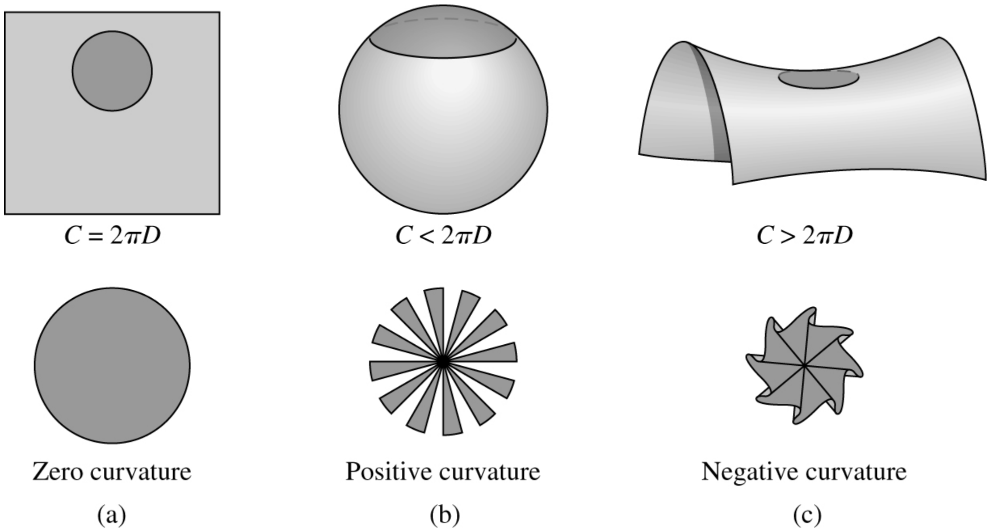


Figure 3.1: The circumference of a circle is equal to the radius $\times 2\pi$ only in a Eucledian geometry. (Reproduced from Carroll & Ostlie's *Modern Astrophysics*).

The distance between two points, $P_1$ and $P_2$ on the surface of a sphere is given by (see Figure 3.2):

$$(d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2 \tag{3.7}$$

But $r = R\,\sin\theta$, so $dr = R\cos\theta\,d\theta$ and

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \tag{3.8}$$

so that:

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2 \tag{3.9}$$

in terms of plane polar coordinates $r$ and $\phi$. More generally, in terms of the curvature K of a two-dimensional surface:

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2 \tag{3.10}$$

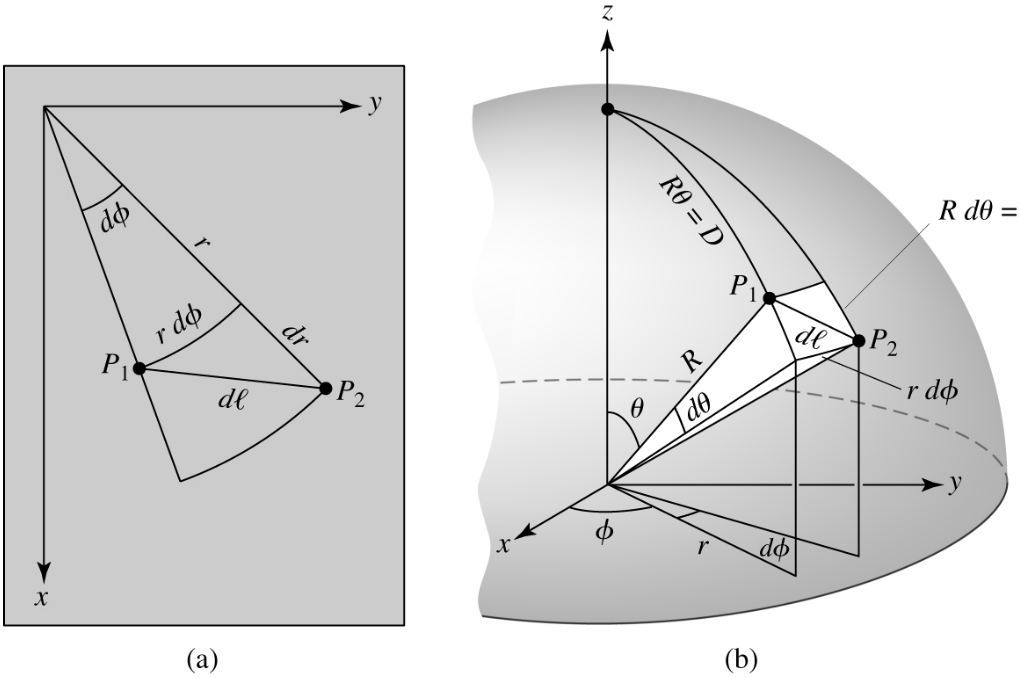


Figure 3.2: (Reproduced from Carroll & Ostlie's *Modern Astrophysics*).

This can be extended to 3-D by changing from polar to spherical coordinates,

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\,\sin\theta\,d\phi)^2 \tag{3.11}$$

where $r$ is now the radial coordinate. Eq. 3.11 shows the effect of the curvature of our three-dimensional Universe on spatial distances.

The final step towards the spacetime metric involves the inclusion of time. By distance we mean the proper distance between two spacetime events that occur simultaneously according to an observer (eq. 3.5). In an expanding universe, the position of two points must be recorded at the same time if their separation is to have any meaning. In an isotropic, homogeneous universe, there is no reason why time should pass at different rates at different locations; thus the temporal term should just be $c\,dt$. The metric then becomes:

$$(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\,\sin\theta\,d\phi)^2 \tag{3.12}$$

and the differential proper distance is just $\Delta\mathcal{L} = \sqrt{-(\Delta s)^2}$ with $dt = 0$.

We can now change our radial coordinate to a comoving coordinate according to eq. 2.4:

$$r(t) = a(t) \cdot x$$

Because the expansion of the Universe affects all of its geometric properties, including its curvature, it is also useful to define the time-dependent curvature in terms of the scale factor and a time independent constant $k$:

$$K(t) = \frac{k}{a^2(t)} \tag{3.13}$$

With these substitutions for $r$ and $K$, we finally arrive at the important Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\left(\frac{dx}{\sqrt{1-kx^2}}\right)^2 + (x\,d\theta)^2 + (x\,\sin\theta\,d\phi)^2\right] \tag{3.14}$$

which is more usually written in the form:

$$\boxed{(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]} \tag{3.15}$$

where, through an annoying change of notation, $r$ now indicates comoving radial distance.

Robertson and Walker independently demonstrated in the mid-1930s that this is the most general metric possible for describing an expanding, homogeneous and isotropic universe.

## 3.2 The Friedmann Equations

The metric evolves according to Einstein's field equations for calculating the geometry of spacetime produced by a given distribution of mass and energy:

$$\mathcal{G}_{\alpha\beta} = \frac{8\pi G}{c^4}\mathcal{T}_{\alpha\beta} \tag{3.16}$$

where $\mathcal{T}_{\alpha\beta}$ is the stress-energy tensor which evaluates the effect of a given distribution of mass and energy on the curvature of spacetime, as described mathematically by Einstein's tensor:

$$\mathcal{G}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \tag{3.17}$$

where $R_{\alpha\beta}$ and $R$ are the Ricci tensor and scalar respectively. The indices $\alpha, \beta$ run over the time coordinate (labelled '0') and the three spatial coordinates.

Making use of the tensor notation, one can write metric equations quite generally:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

where $g_{\alpha\beta}$ is the metric tensor and summation over $\alpha$ and $\beta$ is implied. Orthogonal coordinate systems have diagonal metric tensors and this is all that we need to be concerned with—the metric tensor contains all the information about the intrinsic geometry of spacetime. The components of the Robertson-Walker metric can be written as a diagonal matrix with non-vanishing elements:

$$g_{00} = 1, \qquad g_{11} = -\frac{a^2}{1-kr^2}, \qquad g_{22} = -a^2\, r^2, \qquad g_{33} = -a^2\, r^2\, \sin^2\theta$$

For a comoving observer the time-time component $\mathcal{T}_{00}$ and the space-space component $\mathcal{T}_{11}$ of the stress-energy tensor $\mathcal{T}_{\alpha\beta}$ on the right-hand side of eq 3.16 are:

$$\mathcal{T}_{00} = \rho c^2, \qquad \mathcal{T}_{11} = \frac{pa^2}{1-kr^2} \tag{3.18}$$

where $\rho$ and $p$ are the mass density and the pressure respectively.

On the left-hand side of Einstein's field equations we need $\mathcal{G}_{00}$ and $\mathcal{G}_{11}$ to equate to $\mathcal{T}_{00}$ and $\mathcal{T}_{11}$ respectively. The result of a rather lengthy derivation are:

$$\mathcal{G}_{00} = 3\,(ca)^{-2}\,(\dot{a}^2 + kc^2) \tag{3.19}$$

$$\mathcal{G}_{11} = -c^{-2}\left(2a\ddot{a} + \dot{a}^2 + k\right)\left(1 - kr^2\right)^{-1} \tag{3.20}$$

Substituting (3.18), (3.19) and (3.20) into (3.16), we obtain two distinct dynamical relations for the time evolution of the cosmic scale factor $a(t)$:

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi}{3}\,G\rho && (3.21)\\ 2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= -\frac{8\pi}{c^2}\,Gp && (3.22)\end{aligned}}$$

These equations were derived in 1922 by the Russian physicist and mathematician Alexandr Friedmann, seven years before Hubble's discovery of the universal expansion, at a time when even Einstein did not believe in his own equations because they did not allow the Universe to be static. However, they did not gain general recognition until after Friedmann's death, when they were confirmed by an independent derivation in 1927 by the Belgian cleric Georges Lemaître.

The first Friedmann equation (3.21) shows that the rate of cosmic expansion, $\dot{a}$, increases with the mass density $\rho$ of the universe. Subtracting it from the second Friedmann equation (3.22) we obtain the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3c^2}\,G(\rho c^2 + 3p) \tag{3.23}$$

which shows that the acceleration of the expansion *decreases* with increasing pressure and energy density, whether mass or radiation energy.

At the present time $t_0$, defining $\Omega_0 = \rho_0/\rho_c$ (where $\rho_c$ is the critical density which we defined in eq. 2.12), the Friedmann equation (3.21) takes the form:

$$\dot{a}_0^2 = \frac{8\pi}{3}Ga_0^2\rho_0 - kc^2 = H_0^2 a_0^2\Omega_0 - kc^2 \tag{3.24}$$

This reduces to the Newtonian relation (2.7) if we make the identification:

$$kc^2 = H_0^2 a_0^2(\Omega_0 - 1) \tag{3.25}$$

Equation 3.25 shows explicitly the relation between the curvature parameter $k$ in the Robertson-Walker metric (3.15) and the present-day density parameter $\Omega_0$: to the $k$ values of $+1$, 0, and $-1$ correspond an overcritical density $\Omega_0 > 1$ a critical density $\Omega_0 = 1$ and an undercritical density $0 < \Omega_0 < 1$, respectively.

So, the Friedmann equations are the same, whether derived under Newtonian dynamics or General Relativity. It is only the interpretation of the constant $k$ which is different: in the Newtonian Universe $k$ is related to the mechanical energy of an expanding mass shell by eq. (2.8). In Einstein's Universe it is the present value of the curvature of the Universe (eq. 3.13 with $a = 1$).

## 3.3 The Cosmological Constant

Before the discovery of the cosmic expansion by Hubble in 1929, the universe was thought to be static. This then implies that the scale factor $a \neq f(t)$ but is a constant $a_0$, so that $\dot{a} = \ddot{a} = 0$ (and the age of the universe is infinite).

The two Friedmann equations (3.21 and 3.22) then reduce to:

$$\frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho_0 = -\frac{8\pi}{c^2} Gp_0 \tag{3.26}$$

Note that since $\rho_0$ must be a positive number, $k$ must be $+1$. Note also that this leads to the surprising result that the pressure of matter $p_0$ is negative!

Einstein corrected for this in 1917 by introducing a constant Lorentz-invariant term $\Lambda g_{\alpha\beta}$ into his field equations 3.16:

$$\mathcal{G}_{\alpha\beta} - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta} \tag{3.27}$$

In contrast to the two terms making up the Einstein tensor $\mathcal{G}_{\alpha\beta}$ in eq. 3.17, the $\Lambda g_{\alpha\beta}$ term does not vanish in the limit of flat spacetime.

With this addition, Friedmann's equations take the form:

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi}{3}\,G\rho + \frac{\Lambda c^2}{3} && (3.28) \\ 2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= -\frac{8\pi}{c^2}\,Gp + \Lambda c^2 && (3.29) \end{aligned}}$$

When we write the Friedmann equations in this form, we can see that $\Lambda c^2/8\pi G$ corresponds to an energy density, the *vacuum energy density*, i.e.:

$$\Lambda = \rho_{\rm vac}\,\frac{8\pi G}{c^2}\,. \tag{3.30}$$

Similarly, $\Lambda c^4/8\pi G$ corresponds to a pressure term.[1] A positive value of $\Lambda$ corresponds to a repulsive force counteracting the conventional attractive gravitation, as wanted by Einstein. A negative $\Lambda$ corresponds to an additional attractive force.

From the standpoint of Newtonian cosmology, the additional $\Lambda$ term in Friedmann's equations would result if we added an additional potential energy term:

$$V_\Lambda \equiv -\frac{1}{6}\Lambda mc^2r^2 \tag{3.31}$$

to the right-hand side of eq. 2.5, which would then become:

$$U = T + V + V_\Lambda = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m - \frac{1}{6}\Lambda mc^2r^2\,. \tag{3.32}$$

The force due to this new potential is:

$$\mathbf{F}_\Lambda = -\frac{\partial V_\Lambda}{\partial r}\,\hat{\mathbf{r}} = \frac{1}{3}\Lambda mc^2 r\,\hat{\mathbf{r}} \tag{3.33}$$

which is radially outwards for $\Lambda > 0$.

[1] In some texts, a factor $c^2$ is incorporated in the definition of $\Lambda$—sometimes this is indicated with the lower case letter $\lambda = \Lambda c^2$. As defined here, $\Lambda$ has the units of an inverse area.